**M410**. *Proposé par Matthew Babbitt, étudiant, Albany Area Math Circle, Fort Edward, NY, É-U.*

Un cube d'arête $a$, un tétrahèdre régulier d'arête $b$ et un octahèdre régulier d'arête $c$ ont tous la même surface. Trouver la valeur de $\frac{\sqrt{bc}}{a}$.

**M411**. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

Soit $a$, $b$ et $c$ les côtés du triangle $ABC$. Si

$$2a + 3b + 4c \;=\; 4\sqrt{2a-2} \;+\; 6\sqrt{3b-3} \;+\; 8\sqrt{4c-4} \;-\; 20\,,$$

montrer que $ABC$ est un triangle rectangle.

**M412**. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Si $x$ est un nombre réel, on désigne par $\lfloor x \rfloor$ le plus grand entier plus petit ou égal à $x$, et par $\{x\} = x - \lfloor x \rfloor$ la partie fractionnaire de $x$. Trouver tous les nombres réels $x$ pour lesquels $\lfloor x \rfloor \cdot \{x\} = x$.

# Mayhem Solutions

**M376**. *Proposed by the Mayhem Staff.*

Determine the value of $x$ if $\left(10^{2009}+25\right)^2 - \left(10^{2009}-25\right)^2 = 10^x$.

*Solution by Edin Ajanovic, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina.*

Expanding, we have

$$\begin{aligned}
&\left(10^{2009}+25\right)^2 - \left(10^{2009}-25\right)^2 \\
&\quad= \left[\left(10^{2009}\right)^2 + 2\cdot 25\cdot 10^{2009} + 25^2\right] \\
&\qquad - \left[\left(10^{2009}\right)^2 - 2\cdot 25\cdot 10^{2009} + 25^2\right] \\
&\quad= 4\cdot 25\cdot 10^{2009} \\
&\quad= 100\cdot 10^{2009} \\
&\quad= 10^{2011}\,,
\end{aligned}$$

and so $x = 2011$.

Alternatively, factoring the left side as a difference of squares, we have

$$\begin{aligned}&\left(10^{2009}+25\right)^2-\left(10^{2009}-25\right)^2\\ &= \left[\left(10^{2009}+25\right)-\left(10^{2009}-25\right)\right]\cdot\left[\left(10^{2009}+25\right)+\left(10^{2009}-25\right)\right]\\ &= 50\cdot\left(2\cdot 10^{2009}\right)\\ &= 100\cdot 10^{2009} \;=\; 10^{2011}\,,\end{aligned}$$

and so $x = 2011$, as above.

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; JACLYN CHANG, student, Western Canada High School, Calgary, AB; KATHERINE JANELL EYRE and EMILY HENDRYX, students, Angelo State University, San Angelo, TX, USA; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON.*

## M377. *Proposed by the Mayhem Staff.*

An arithmetic sequence consists of $9$ positive integers. The sum of the terms in the sequence is greater than $200$ and less than $220$. If the second term in the sequence is $12$, determine the sequence.

*Solution by Katherine Janell Eyre and Emily Hendryx, students, Angelo State University, San Angelo, TX, USA.*

Suppose that $a$ is the first term in the sequence and $d$ is the difference between successive terms. Then using formulae for arithmetic sequences and series, the $n^{\text{th}}$ term can be written as $a_n = a + (n-1)d$ and the sum of the first $n$ terms can be written as $S_n = \frac{n}{2}\bigl(2a + (n-1)d\bigr)$. Hence,

$$S_9 \;=\; \frac{9}{2}(2a+8d) \;=\; 9(a+4d)\,.$$

Also, we know that $a_2 = 12 = a + d$, from which we obtain $d = 12 - a$. Combining these, we get

$$S_9 \;=\; 9(a+4d) \;=\; 9\bigl(a+4(12-a)\bigr) \;=\; 9(48-3a) \;=\; 27(16-a)\,.$$

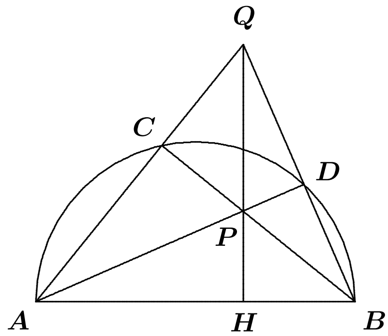
Since $200 < S_9 < 220$, we get $200 < 27(16-a) < 220$ which reduces to $7\frac{11}{27} < 16 - a < 8\frac{4}{27}$, or $7\frac{23}{27} < a < 8\frac{16}{27}$. Since $a$ has to be a positive integer, we have $a = 8$, and so $d = 12 - a = 4$. We can then check that the resulting sequence satisfies all the requirements. Therefore, the sequence is $8$, $12$, $16$, $20$, $24$, $28$, $32$, $36$, $40$.

*Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT,* *student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; JACLYN CHANG, student, Western Canada High School, Calgary, AB; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incomplete solution submitted.*

## M378. *Proposed by the Mayhem Staff.*

Points $C$ and $D$ are chosen on the semicircle with diameter $AB$ so that $C$ is closer to $A$. Segments $CB$ and $DA$ intersect at $P$; segments $AC$ and $BD$ extended intersect at $Q$. Prove that $QP$ extended is perpendicular to $AB$.

*Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.*




Since $AB$ is a diameter, then $\angle ACB = \angle ADB = 90°$. Thus, $\angle QCP = \angle QDP = 90°$, so quadrilateral $CPDQ$ is cyclic since a pair of opposite angles adds to $180°$. Note that $ACDB$ is also cyclic.

Let $H$ be the intersection of $PQ$ extended and $AB$. Then we have $\angle ABC = \angle ADC$, since each is subtended by the same arc. As well, $\angle PDC = \angle CQP$ since $CPDQ$ is a cyclic quadrilateral. Therefore, $\angle ADC = \angle PDC = \angle CQP = \angle AQH$. Hence, $\angle ABC = \angle AQH$.

But $\angle CAB = \angle QAH$ is a common angle in $\triangle ABC$ and $\triangle QAB$, and so $\triangle AQH$ is similar to $\triangle ABC$. Therefore, $\angle AHQ = \angle ACB = 90°$, and so $QP$ extended is perpendicular to $AB$.

*Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MATTHEW BABBITT, student, Albany Area Math Circle, Fort Edward, NY, USA; LYLE BERSTEIN, student, Roslyn High School, Roslyn, NY, USA; ANTONIO GODOY TOHARIA, Madrid, Spain; R. LAUMEN, Deurne, Belgium; HUGO LUYO SÁNCHEZ, Pontificia Universidad Católica del Peru, Lima, Peru; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania.*

*Most solvers noted that $AD$ and $BC$ are altitudes of $\triangle ABQ$. Thus, their point of intersection, $P$, is the orthocentre of the triangle and so the line segment through $Q$ and $P$ must be the third altitude of the triangle, and so is perpendicular to $AB$, as required.*

## M379. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB.*

The integers $27+C$, $555+C$, and $1371+C$ are all perfect squares, the square roots of which form an arithmetic sequence. Determine all possible values of $C$.

*Solution by Matthew Babbitt, student, Albany Area Math Circle, Fort Edward, NY, USA.*

Since the three given expressions have square roots in arithmetic sequence, then we may suppose that the square roots of $27+C$, $555+C$, and $1371+C$ are $a-d$, $a$, and $a+d$, respectively, for some $a$ and $d$. Thus, $(a-d)^2 = 27+C$, $a^2 = 555+C$, and $(a+d)^2 = 1371+C$.

Note that

$$\begin{aligned}(a+d)^2+(a-d)^2-2a^2 &= \left(a^2+2ad+d^2\right)+\left(a^2-2ad+d^2\right)-2a^2 \\ &= 2d^2\,,\end{aligned}$$

and so

$$(1371+C)\ +\ (27+C)\ -\ 2(555+C)\ =\ 2d^2\,,$$

or $288 = 2d^2$, or $d^2 = 144$, which yields $d = \pm 12$.

If $d = 12$, then $(a+d)^2 = 1371+C$ and $(a+d)^2 = a^2+2ad+d^2 = 555+C+24a+144$. Combining these equations yields $24a = 672$, or $a = 28$. Thus, $C = a^2-555 = 28^2-555 = 229$.

If $d = -12$, then $(a+d)^2 = 1371+C$ and $(a+d)^2 = a^2-2ad+d^2 = 555+C-24a+144$. Combining these equations yields $-24a = 672$, or $a = -28$. Thus, $C = a^2-555 = (-28)^2-555 = 229$.

In either case, $C = 229$. We can check by substitution that $C = 229$ has the desired properties.

*Also solved by EDIN AJANOVIC, student, First Bosniak High School, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ANTONIO GODOY TOHARIA, Madrid, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; R. LAUMEN, Deurne, Belgium; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There was one incorrect solution submitted.*

**M380**. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Triangle $ABC$ is right-angled at $C$ and has $BC = a$ and $CA = b$, with $a \geq b$. Squares $ABDE$, $BCFG$, and $CAHI$ are drawn externally to triangle $ABC$. The lines through $FI$ and $EH$ intersect at $P$, the lines through $FI$ and $DG$ intersect at $Q$, and the lines through $DG$ and $EH$ intersect at $R$. If triangle $PQR$ is right-angled, determine the value of $\dfrac{b}{a}$.

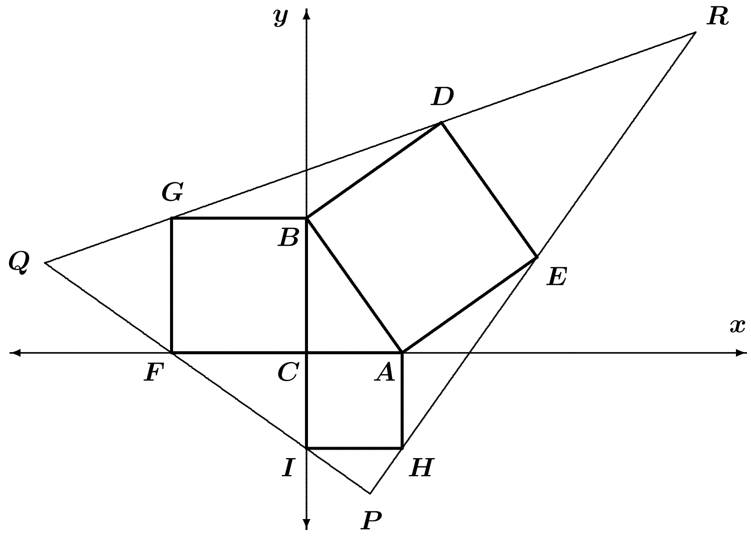
*Composite of solutions by George Apostolopoulos, Messolonghi, Greece and Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.*

We coordinatize the diagram on the following page by placing $C$ at $(0,0)$, $B$ at $(0,a)$ and $A$ at $(b,0)$. Note that $a \geq b > 0$.

Since $BCFG$ is a square external to $\triangle ABC$, then it has side length $a$ and has sides parallel to the axes, so the coordinates of $G$ are $(-a, a)$ and the coordinates of $F$ are $(-a, 0)$.

Similarly, since $CAHI$ is a square external to $\triangle ABC$, the coordinates of $H$ are $(b, -b)$ and the coordinates of $I$ are $(0, -b)$.

Since $ABDE$ is a square external to $\triangle ABC$ and the vector $\overrightarrow{AB}$ equals $(b, -a)$, then the vectors $\overrightarrow{BD}$ and $\overrightarrow{AE}$ are both equal to $(a, b)$ (because they are perpendicular to $AB$ and equal in length). Thus, the coordinates of $D$ are $(a, a+b)$ and the coordinates of $E$ are $(a+b, b)$.




The slope of $QP$ equals the slope of $FI$. The coordinates of $F$ are $(-a, 0)$ and the coordinates of $I$ are $(0, -b)$, so the slope of $QP$ is $-\frac{b}{a}$.

The slope of $QR$ equals the slope of $GD$. The coordinates of $G$ are $(-a, a)$ and the coordinates of $D$ are $(a, a+b)$, so the slope of $QR$ is $\frac{b}{2a}$.

The slope of $PR$ equals the slope of $HE$. The coordinates of $H$ are $(b, -b)$ and the coordinates of $E$ are $(a+b, b)$, so the slope of $PR$ is $\frac{2b}{a}$.

For two line segments to be perpendicular, the product of their slopes must equal $-1$. Since the slopes of $QR$ and $PR$ are both positive, then these segments are not perpendicular and the angle at $R$ is not a right angle.

If there was a right angle at $Q$, then $QR$ and $QP$ would be perpendicular, or $-\frac{b}{a} \cdot \frac{b}{2a} = -1$, or $b^2 = 2a^2$, contradicting the fact that $a \geq b > 0$.

Thus, if $\triangle PQR$ has a right angle, then it can only occur at $P$. In that case, the segments $QP$ and $PR$ are perpendicular, so $-\frac{b}{a} \cdot \frac{2b}{a} = -1$, or $\frac{b^2}{a^2} = \frac{1}{2}$. Since $a \geq b > 0$, then $\frac{b}{a} = \frac{1}{\sqrt{2}}$.

*Also solved by RICARD PEIRÓ, IES "Abastos", Valencia, Spain.*