

**1999**  
**SOLUTIONS**

Most of the solutions to the problems of the 1999 CMO presented below are taken from students' papers. Some minor editing has been done - unnecessary steps have been eliminated and some wording has been changed to make the proofs clearer. But for the most part, the proofs are as submitted.

**Solution to Problem 1 – Adrian Chan, Upper Canada College, Toronto, ON**

Rearranging the equation we get  $4x^2 + 51 = 40[x]$ . It is known that  $x \geq [x] > x - 1$ , so

$$\begin{aligned}4x^2 + 51 = 40[x] &> 40(x - 1) \\4x^2 - 40x + 91 &> 0 \\(2x - 13)(2x - 7) &> 0\end{aligned}$$

Hence  $x > 13/2$  or  $x < 7/2$ . Also,

$$\begin{aligned}4x^2 + 51 = 40[x] &\leq 40x \\4x^2 - 40x + 51 &\leq 0 \\(2x - 17)(2x - 3) &\leq 0\end{aligned}$$

Hence  $3/2 \leq x \leq 17/2$ . Combining these inequalities gives  $3/2 \leq x < 7/2$  or  $13/2 < x \leq 17/2$ .

**CASE 1:**  $3/2 \leq x < 7/2$ .

For this case, the possible values for  $[x]$  are 1, 2 and 3.

If  $[x] = 1$  then  $4x^2 + 51 = 40 \cdot 1$  so  $4x^2 = -11$ , which has no real solutions.

If  $[x] = 2$  then  $4x^2 + 51 = 40 \cdot 2$  so  $4x^2 = 29$  and  $x = \frac{\sqrt{29}}{2}$ . Notice that  $\frac{\sqrt{16}}{2} < \frac{\sqrt{29}}{2} < \frac{\sqrt{36}}{2}$  so  $2 < x < 3$  and  $[x] = 2$ .

If  $[x] = 3$  then  $4x^2 + 51 = 40 \cdot 3$  and  $x = \sqrt{69}/2$ . But  $\frac{\sqrt{69}}{2} > \frac{\sqrt{64}}{2} = 4$ . So, this solution is rejected.

**CASE 2:**  $13/2 < x \leq 17/2$ .

For this case, the possible values for  $[x]$  are 6, 7 and 8.

If  $[x] = 6$  then  $4x^2 + 51 = 40 \cdot 6$  so  $x = \frac{\sqrt{189}}{2}$ . Notice that  $\frac{\sqrt{144}}{2} < \frac{\sqrt{189}}{2} < \frac{\sqrt{196}}{2}$  so  $6 < x < 7$  and  $[x] = 6$ .

If  $[x] = 7$  then  $4x^2 + 51 = 40 \cdot 7$  so  $x = \frac{\sqrt{229}}{2}$ . Notice that  $\frac{\sqrt{196}}{2} < \frac{\sqrt{229}}{2} < \frac{\sqrt{256}}{2}$  so  $7 < x < 8$  and  $[x] = 7$ .

If  $[x] = 8$  then  $4x^2 + 51 = 40 \cdot 8$  so  $x = \frac{\sqrt{269}}{2}$ . Notice that  $\frac{\sqrt{256}}{2} < \frac{\sqrt{269}}{2} < \frac{\sqrt{324}}{2}$  so  $8 < x < 9$  and  $[x] = 8$ .

The solutions are  $x = \frac{\sqrt{29}}{2}, \frac{\sqrt{189}}{2}, \frac{\sqrt{229}}{2}, \frac{\sqrt{269}}{2}$ .

(Editor: Adrian then checks these four solutions.)

**Solution 1 to Problem 2 – Keon Choi, A.Y. Jackson S.S., North York, ON**

Let  $D$  and  $E$  be the intersections of  $BC$  and extended  $AC$  respectively with the circle.

Since  $CO \parallel AB$  (because both the altitude and the radius are 1)  $\angle BCO = 60^\circ$  and therefore  $\angle ECO = 180^\circ - \angle ACB - \angle BCO = 60^\circ$ .

Since a circle is always symmetric in its diameter and line  $CE$  is reflection of line  $CB$  in  $CO$ , line segment  $CE$  is reflection of line segment  $CB$ .

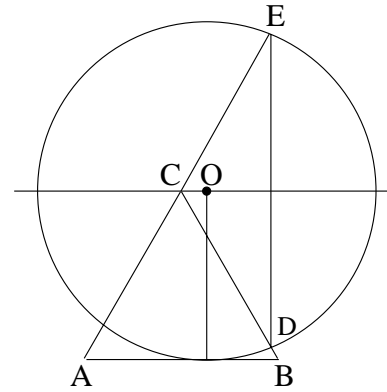
Therefore  $CE = CD$ .

Therefore  $\triangle CED$  is an isosceles.

Therefore  $\angle CED = \angle CDE$  and  $\angle CED + \angle CDE = \angle ACB = 60^\circ$ .

$\angle CED = 30^\circ$  regardless of the position of centre  $O$ . Since  $\angle CED$  is also the angle subtended from the arc inside the triangle, if  $\angle CED$  is constant, the arc length is also constant.

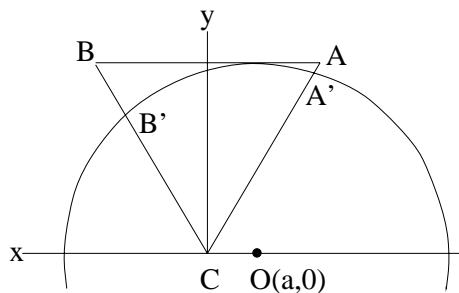
*Editor's Note:* This proof has had no editing.



**Solution 2 to Problem 2 – Jimmy Chui, Earl Haig S.S., North York, ON**

Place  $C$  at the origin, point  $A$  at  $(\frac{1}{\sqrt{3}}, 1)$  and point  $B$  at  $(-\frac{1}{\sqrt{3}}, 1)$ . Then  $\triangle ABC$  is equilateral with altitude of length 1.

Let  $O$  be the center of the circle. Because the circle has radius 1, and since it touches line  $AB$ , the locus of  $O$  is on the line through  $C$  parallel to  $AB$  (since  $C$  is length 1 away from  $AB$ ), i.e., the locus of  $O$  is on the  $x$ -axis.



Let point  $O$  be at  $(a, 0)$ . Then  $-\frac{1}{\sqrt{3}} \leq a \leq \frac{1}{\sqrt{3}}$  since we have the restriction that the circle rolls along  $AB$ .

Now, let  $A'$  and  $B'$  be the intersection of the circle with  $CA$  and  $CB$  respectively. The equation of  $CA$  is  $y = \sqrt{3}x$ ,  $0 \leq x \leq \frac{1}{\sqrt{3}}$ , of  $CB$  is  $y = -\sqrt{3}x$ ,  $-\frac{1}{\sqrt{3}} \leq x \leq 0$ , and of the circle is  $(x - a)^2 + y^2 = 1$ .

We solve for  $A'$  by substituting  $y = \sqrt{3}x$  into  $(x - a)^2 + y^2 = 1$  to get  $x = \frac{a \pm \sqrt{4 - 3a^2}}{4}$ .

Visually, we can see that solutions represent the intersection of  $AC$  extended and the circle, but we are only concerned with the greater  $x$ -value – this is the solution that is on  $AC$ , not on  $AC$  extended. Therefore

$$x = \frac{a + \sqrt{4 - 3a^2}}{4}, \quad y = \sqrt{3} \left( \frac{a + \sqrt{4 - 3a^2}}{4} \right).$$

Likewise we solve for  $B'$ , but we take the lesser  $x$ -value to get

$$x = \frac{a - \sqrt{4 - 3a^2}}{4}, \quad y = -\sqrt{3} \left( \frac{a + \sqrt{4 - 3a^2}}{4} \right).$$

Let us find the length of  $A'B'$ :

$$\begin{aligned} |A'B'|^2 &= \left( \frac{a + \sqrt{4 - 3a^2}}{4} - \frac{a - \sqrt{4 - 3a^2}}{4} \right)^2 + \left( \left( \sqrt{3} \frac{a + \sqrt{4 - 3a^2}}{4} \right) - \left( -\sqrt{3} \frac{a - \sqrt{4 - 3a^2}}{4} \right) \right)^2 \\ &= \frac{4 - 3a^2}{4} + 3 \frac{a^2}{4} \\ &= 1 \end{aligned}$$

which is independent of  $a$ .

Consider the points  $O$ ,  $A'$  and  $B'$ .  $\triangle OA'B'$  is an equilateral triangle (because  $A'B' = OA' = OB' = 1$ ). Therefore  $\angle A'OB' = \frac{\pi}{3}$  and arc  $A'B' = \frac{\pi}{3}$ , a constant.

### Solution to Problem 3 – Masoud Kamgarpour, Carson S.S., North Vancouver, BC

Note that  $n = 1$  is a solution. For  $n > 1$  write  $n$  in the form  $n = P_1^{\alpha_1} P_2^{\alpha_2} \dots P_m^{\alpha_m}$  where the  $P_i$ 's,  $1 \leq i \leq m$ , are distinct prime numbers and  $\alpha_i > 0$ . Since  $d(n)$  is an integer,  $n$  is a perfect square, so  $\alpha_i = 2\beta_i$  for integers  $\beta_i > 0$ .

Using the formula for the number of divisors of  $n$ ,

$$d(n) = (2\beta_1 + 1)(2\beta_2 + 1) \dots (2\beta_m + 1)$$

which is an odd number. Now because  $d(n)$  is odd,  $(d(n))^2$  is odd, therefore  $n$  is odd as well, so  $P_i \geq 3, 1 \leq i \leq m$ . We get

$$P_1^{\alpha_1} \cdot P_2^{\alpha_2} \dots P_m^{\alpha_m} = [(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_m + 1)]^2$$

or using  $\alpha_i = 2\beta_i$

$$P_1^{\beta_1} P_2^{\beta_2} \dots P_m^{\beta_m} = (2\beta_1 + 1)(2\beta_2 + 1) \dots (2\beta_m + 1).$$

Now we prove a lemma:

Lemma:  $P^t \geq 2t + 1$  for positive integers  $t$  and  $P \geq 3$ , with equality only when  $P=3$  and  $t=1$ .

Proof: We use mathematical induction on  $t$ . The statement is true for  $t = 1$  because  $P \geq 3$ . Now suppose  $P^k \geq 2k + 1, k \geq 1$ ; then we have

$$P^{k+1} = P^k \cdot P \geq P^k(1 + 2) > P^k + 2 \geq (2k + 1) + 2 = 2(k + 1) + 1$$

Thus  $P^t \geq 2t + 1$  and equality occurs only when  $P = 3$  and  $t = 1$ .

Let's say  $n$  has a prime factor  $P_k > 3$ ; then (by the lemma)  $P_k^{\beta_k} > 2\beta_k + 1$  and we have  $P_1^{\beta_1} \dots P_m^{\beta_m} > (2\beta_1 + 1) \dots (2\beta_m + 1)$ , a contradiction.

Therefore, the only prime factor of  $n$  is  $P = 3$  and we have  $3^\alpha = 2\alpha + 1$ . By the lemma  $\alpha = 1$ .

The only positive integer solutions are 1 and 9.

### Solution 1 to Problem 4 – David Nicholson, Fenelon Falls S.S., Fenelon Falls, ON

Without loss of generality let  $a_1 < a_2 < a_3 \dots < a_8$ .

Assume that there is no such integer  $k$ . Let's just look at the seven differences  $d_i = a_{i+1} - a_i$ . Then amongst the  $d_i$  there can be at most two 1s, two 2s, and two 3s, which totals to 12.

Now  $16 = 17 - 1 \geq a_8 - a_1 = d_1 + d_2 + \dots + d_7$  so the seven differences must be 1,1,2,2,3,3,4.

Now let's think of arranging the differences 1,1,2,2,3,3,4. Note that the sum of consecutive differences is another difference. (Eg  $d_1 + d_2 = a_3 - a_1$ ,  $d_1 + d_2 + d_3 = a_4 - a_1$ )

We can't place the two 1s side by side because that will give us another difference of 2. The 1s can't be beside a 2 because then we have three 3s. They can't both be beside a 3 because then we have three 4s! So we must have either 1, 4, -, -, -, 3, 1 or 1, 4, 1, 3, -, -, - (or their reflections).

In either case we have a 3,1 giving a difference of 4 so we can't put the 2s beside each other. Also we can't have 2,3,2 because then (with the 1,4) we will have three 5s. So all cases give a contradiction.

Therefore there will always be three differences equal.

A set of seven numbers satisfying the criteria are  $\{1, 2, 4, 7, 11, 16, 17\}$ . (Editor: There are many such sets)

### Solution 2 to Problem 4 – The CMO committee

Consider all the consecutive differences (ie,  $d_i$  above) as well as the differences  $b_i = a_{i+2} - a_i$ ,  $i = 1 \dots 6$ . Then the sum of these thirteen differences is  $2 \cdot (a_8 - a_1) + (a_7 - a_2) \leq 2(17-1) + (16-2) = 46$ . Now if no difference occurs more than twice, the smallest the sum of the thirteen differences can be is  $2 \cdot (1 + 2 + 3 + 4 + 5 + 6) + 7 = 49$ , giving a contradiction.

### Solution 1 to Problem 5 – The CMO committee

Let  $f(x, y, z) = x^2y + y^2z + z^2x$ . We wish to determine where  $f$  is maximal. Since  $f$  is cyclic, without loss of generality we may assume that  $x \geq y, z$ . Since

$$\begin{aligned} f(x, y, z) - f(x, z, y) &= x^2y + y^2z + z^2x - x^2z - z^2y - y^2x \\ &= (y - z)(x - y)(x - z), \end{aligned}$$

we may also assume  $y \geq z$ . Then

$$\begin{aligned} f(x + z, y, 0) - f(x, y, z) &= (x + z)^2y - x^2y - y^2z - z^2x \\ &= z^2y + yz(x - y) + xz(y - z) \geq 0, \end{aligned}$$

so we may now assume  $z = 0$ . The rest follows from the arithmetic-geometric mean inequality:

$$f(x, y, 0) = \frac{2x^2y}{2} \leq \frac{1}{2} \left( \frac{x + x + 2y}{3} \right)^3 = \frac{4}{27}$$

Equality occurs when  $x = 2y$ , hence at  $(x, y, z) = (\frac{2}{3}, \frac{1}{3}, 0)$ . (As well as  $(0, \frac{2}{3}, \frac{1}{3})$  and  $(\frac{1}{3}, 0, \frac{2}{3})$ .)

### Solution 2 to Problem 5 – The CMO committee

With  $f$  as above, and  $x \geq y, z$

$$f\left(x + \frac{z}{2}, y + \frac{z}{2}, 0\right) - f(x, y, z) = yz(x - y) + \frac{xz}{2}(x - z) + \frac{z^2y}{4} + \frac{z^3}{8}$$

so we may assume that  $z = 0$ . The rest follows as for solution 1.

## GRADERS' REPORT

Each question was worth a maximum of 7 marks. Every solution on every paper was graded by two different markers. If the two marks differed by more than one point, the solution was reconsidered until the difference resolved. If the two marks differed by one point, the average was used in computing the total score.

The various grades assigned to each solution are displayed below, as a percentage.

MARKS	#1	#2	#3	#4	#5
0	7.6	45.6	18.4	38.6	51.3
1	13.9	15.8	43.0	0.0	32.3
2	12.0	12.7	15.2	41.8	13.9
3	5.7	2.5	5.1	7.6	2.5
4	4.4	1.3	8.9	4.4	0.0
5	8.9	2.5	3.2	5.7	1.3
6	8.9	0.0	1.3	1.9	0.0
7	39.9	20.9	6.3	1.3	0.0

**PROBLEM 1** The aim of the question was to give the competitors an encouraging start (it was not a give away!). Over half of the students had good scores of 5, 6 or 7.

The general approach was to find bounds for  $x$  and then to find the exact value for  $x$  by substituting in the resulting possible values of  $[x]$ . Depending on how the bounds were determined, this meant checking 6 - 10 different cases.

Points were lost for not adequately verifying the bounds on  $x$ . For example, 2 points were deducted for assuming, without proof, that  $4x^2 + 51 > 40[x]$  for  $x \geq 9$ .

**PROBLEM 2** Many competitors saw that the key here is to prove that the angle subtended by the arc at its centre is constant, namely  $\pi/3$ . In all, 16 students managed a complete proof. Most attempted an analytic solution – indeed, the problem is nearly routine if one chooses coordinates wisely and later on notes that two such  $x$ -coordinates are roots of the same quadratic. A few students used trigonometry, namely the law of sines on a couple of useful triangles. Two students found essentially the same synthetic solution, which is very elegant.

**PROBLEM 3** Most competitors determined by direct calculation that  $n = 1$  and  $n = 9$  are solutions. The difficulty was to show that these are the only solutions, which boils down to proving that  $p^k \geq 2k + 1$  for all primes  $p > 2$  and all  $k > 0$  with equality only for  $k = 1$  and  $p = 3$ . This can be done by induction or by calculus. Only 5 students obtained perfect marks.

### PROBLEM 4

Many students found a specific set of seven integers such that the equation did not have three different solutions. This earned two points. (One student found such a set with maximum value 14. A maximum value of 13 is not possible.)

Only eight competitors received high marks on the question (5, 6, or 7), and only one student scored a perfect 7. All of the successful solvers considered differences of consecutive integers, showing that they must be 1, 1, 2, 2, 3, 3, and 4, and then showed that every ordering of these differences led to at least three repetitions of the same value. Most competitors recognized that the 1s could not be together, nor could they be beside a 2. They then proceed by considering all such possible arrangements, which often resulted in close to a dozen cases (depending on how the the cases were handled.) David Nicholson was the most efficient at pruning the cases. (See Solution 1 to Problem 4.) Most students failed to consider one or two (easily dismissed) cases, hence lost 1 or 2 points.

A number of the contestants attempted to solve the problem by examining the odd-even character of the set of eight integers, counting how many of the differences were odd or even, and using the pigeon-hole principle. Although this approach looked promising, no one was able to handle the case that 3 of the integers were of one parity, and 5 were of the other parity.

**PROBLEM 5** No students received full marks for this problem. One student received 5 marks for a proof that had minor errors. This proof was by Calculus. The committee was aware that the problem could be solved using Calculus but (erroneously) thought it unlikely high school students would attempt such a solution. Many students received 1 point for “guessing” that  $(\frac{2}{3}, \frac{1}{3}, 0)$ ,  $(0, \frac{2}{3}, \frac{1}{3})$  and  $(\frac{1}{3}, 0, \frac{2}{3})$  are where equality occurred. Some students received a further point for verifying the inequality on the boundary of the region.